the corresponding published values in Ref. 4 by less than 5 in the fourth decimal place at each of the 36 tabulated points with the calculated value $f''(0)^{(5)} = 1.2326740$ as compared with 1.232588.

The sample calculations other than for the case of $\beta = 1$ were carried out for the cases of $\beta = 0.5$ and 0 using the same initial approximations and the same values of η_e, h and ε as for the case of $\beta = 1$. The converged solutions were obtained after 4 and 8 iterations, respectively.

Unfortunately, the present method cannot be applied to a problem in which the uniqueness of the solution is not assured. For such a problem, special account of the asymptotic behavior of the solution at the outer edge of the boundary layer must be taken and an improved version of quasi-linearization by Libby and Chen[3] seems to be the best of the existing methods. The present method, despite the restriction, seems to provide another useful technique for treating the two-point boundary value problem and is also applicable to the numerical analysis of nonsimilar flows by a difference-differential scheme, such as has been developed by Smith and Clutter.[5]

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Variations of Eigenvalues and Eigenfunctions in Continuum Mechanics

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I. Introduction

THE main procedure of a typical design problem usually consists of repeated modifications of design parameters and the investigation of the system response for each set of these parameters to achieve certain system performance, corresponding to favorite design. A more ambitious aim may be to achieve the best design which optimizes the response of the system. Whatever the goal of design may be, the repeated analysis is, in nature, costly and, hence, there exists an increasing need for the prediction of the system response, due to changes in design parameters, in a more efficient way. The literature contains some attempts to the end of prediction of the response of a system with new set of design parameters from the response of the old system. (See for example Refs. 1–6.) Attempts of this sort have been, so far, directed towards the study of discrete models according to which the system can be designed and synthesized by a set of numbers, the so called "design parameters." A more realistic and refined idea of engineering design would be the characterization of systems by a set of "design functions." Such a consideration finds a vital importance in the true optimization problems in which the perfect design can only be achieved through the investigation of systems with distributed properties. With the above notions in mind, the present work is aimed at the prediction of the change in response of continuous systems caused by the variations of design functions. We shall obtain a set of relations which give the firsit-order variations of the eigenvalues and eigenfunctions of the new design in terms of these quantities belonging to the previous design. As illustrative examples, several classes of problems in vibration, stability and optimization are treated.

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II. Variation of Eigenvalues

Consider the following continuous system of eigenvalue problem:

$$\mathbf{LY} + \lambda \mathbf{MY} = 0 \tag{1}$$

where $\mathbf{Y}$ is the state vector and $\mathbf{L}$ and $\mathbf{M}$ are, in matrix form, two linear differential operators and embody the functions $h_i(x)$ characterizing the system, that is

$$\mathbf{L} = \mathbf{L}[h_i(x)], \qquad \mathbf{M} = \mathbf{M}[h_i(x)] \tag{2}$$

these field equations are to be accompanied by a set of appropriate boundary conditions. The above system is allowed to be nonself adjoint in the general case. Suppose that λ_m and $\mathbf{Y}_m$ $(m = 1, 2, \ldots)$ are the eigenvalues and the corresponding eigenfunctions of the system and let $\mathbf{Y}_n{}^*$ be the eigenfunction of the associated adjoint system. The biorthogonality condition of Eq. (1) and its adjoint can be expressed as[7]

$$\int_D \mathbf{Y}_n{}^{*\prime}\mathbf{MY}_m dD = (\mathbf{Y}_n{}^*, \mathbf{MY}_m) = \delta_{mn} = \begin{vmatrix} 0 & m \perp n \\ 1 & m = n \end{vmatrix} \tag{3}$$

wherein D is the domain of integration and $\mathbf{Y}_n{}^{*\prime}$ is the transpose of $\mathbf{Y}_n{}^*$.

If the functions $h_i(x)$ characterizing the system are varied, the characteristic response of the system such as its eigenvalues and eigenfunctions will also change. To the end of determination of the variation of these quantities, denoted by $\delta\lambda_m$ and $\delta\mathbf{Y}_m$, due to variation of $h_i(x)$, denoted by $\delta h_i(x)$, we do as follows:

The Rayleigh quotient corresponding to the eigenfunction $\mathbf{Y}_m$ of the system[1] is

$$(\mathbf{Y}_m{}^*, \mathbf{LY}_m) + \lambda_m(\mathbf{Y}_m{}^*, \mathbf{MY}_m) = 0 \tag{4}$$

If we take the first variation of both sides of Eq. (4), with the help of relations (1) and (3), we get

$$\delta\lambda_m = -[(\mathbf{Y}_m{}^*, \delta\mathbf{LY}_m) + \lambda_m(\mathbf{Y}_m{}^*, \delta\mathbf{MY}_m)] \tag{5}$$

Note that the variation symbol, δ, in front of a letter implies the variation of that letter only and not the subsequent letters. Equation (5) yields the variation of the eigenvalue λ_m, due to variation of $h_i(x)$, in terms of already known eigenvalues and eigenfunctions of the original problem with unperturbed $h_i(x)$.

III. Variation of Eigenfunctions

If we take the first variation of both sides of Eq. (1), we obtain

$$\delta\mathbf{LY}_m + L\,\delta\mathbf{Y}_m + \delta\lambda_m\mathbf{MY}_m + \lambda_m\,\delta\mathbf{MY}_m + \lambda_m\mathbf{M}\,\delta\mathbf{Y}_m = 0 \tag{6}$$

Now, considering the fact that the eigenfunctions $\{\mathbf{Y}_m\}$ form a complete set we can expand the variation of $\mathbf{Y}_m$ in terms of these eigenfunctions, so we have

$$\delta\mathbf{Y}_m = \sum_j a_{mj}\mathbf{Y}_j \tag{7}$$

Substituting the above expansion into Eq. (6) and taking the inner product, according to Eq. (3), of both sides of the resulting equation with $\mathbf{Y}_n{}^*$, we obtain

$$(\mathbf{Y}_n{}^*, \delta\mathbf{LY}_m) + (\mathbf{Y}_n{}^*, \sum_j a_{mj}\mathbf{LY}_j) + \delta\lambda_m(\mathbf{Y}_n{}^*, \mathbf{MY}_m) + \lambda_m(\mathbf{Y}_n{}^*, \delta\mathbf{MY}_m) + \lambda_m(\mathbf{Y}_n{}^*, \sum_j a_{mj}\mathbf{MY}_j) = 0 \tag{8}$$

Finally the substitution of j for n and the utilization of Eqs. (1) and (3) yields

$$a_{mj} = [-1/(\lambda_m - \lambda_j)][(Y_j{}^*, \delta LY_m) + \lambda_m(Y_j{}^*, \delta MY_m)] \quad m \neq j \tag{9}$$

to get a_{mm} we use Eq. (3), for the case of self-adjoint systems and we find

$$a_{mm} = -\tfrac{1}{2}(\mathbf{Y}_m, \delta\mathbf{M}\mathbf{Y}_m) \tag{10}$$

Hence, the relation (7) together with the expressions (9) and (10) gives the variation of the eigenfunction, corresponding to eigenvalue λ_m, in terms of initially calculated quantities.

If the system is nonself adjoint, a_{mm} is determined by observing that we can normalize, to one, one of $\mathbf{Y}_m$'s at a specific point before and after variational procedure; if we do so and using Eq. (7) we get

$$\delta Y_m(x_o) = \sum_j a_{mj} Y_j(x_o) \tag{11}$$

from which we obtain

$$a_{mm} = -\sum_{m \neq j} a_{mj} \mathbf{Y}_j(x_o)$$

IV. Applications

Example A)—Sturm-Liouville Problems

As an illustrating specific example consider the class of Sturm-Liouville systems resulting from the following functional [8]:

$$\lambda = \int_a^b [p(x)y'^2 - q(x)y^2]\,dx \Big/ \int_a^b r(x)y^2\,dx \tag{12}$$

The aim is the calculation of changes in eigenvalues and eigenfunctions of this class of problems arising from the change of functions $p(x)$, $q(x)$, and $r(x)$ if we denote an arbitrary change of these functions by $\delta p(x)$, $\delta q(x)$, and $\delta r(x)$, respectively, and by designating the associated eigenvalue and eigenfunctions' changes by $\delta\lambda_m$ and δy_m, and utilize Eq. (5) we get

$$\delta\lambda_m = -\int_a^b (\delta p y_m'^2 + \delta q y_m^2 + \lambda_m \delta r y_m^2)\,dx \tag{13}$$

and by using Eqs. (9) and (10) we obtain

$$a_{mj} = -\frac{1}{\lambda_m - \lambda_j}\int_a^b (\delta p y_m' y_j' + \delta q y_m y_j + \lambda_m \delta r y_m y_j)\,dx \tag{14}$$

$$a_{mm} = -\frac{1}{2}\int_a^b \delta_r y_m^2\,dx$$

As a more specific example of a physical problem governed by the above class of equations we consider the longitudinal vibrations of a fixed-free rod. Starting with a uniform and homogeneous rod we would like to investigate the effect of the change in rod density on its natural frequencies and normal modes of vibrations. The natural frequencies and natural modes of the unperturbed (uniform) rod are

$$\lambda_m = \tfrac{1}{4}(m\pi)^2, \quad Y_m = \sin(m\pi/2)x, \quad m = 1, 3, 5, \ldots \tag{15}$$

if the variation in density is denoted by the nondimensionalized function $\rho(x)$ the governing equations of time harmonic motion of density-perturbed rod is

$$d^2Y/dx^2 + [1 + \rho(x)]\lambda Y = 0 \tag{16}$$

where x is the length parameter, Y is the longitudinal displacement, and λ is the square of natural frequency; all in nondimensionalized quantities. According to Eq. (13) the variation in the natural frequency due to variation of the density is

$$\delta\lambda_m = -\int_o^1 \lambda_m \rho(x) Y_m^2\,dx \tag{17}$$

Let, for the sake of illustration, $\rho(x) = \varepsilon x$, then Eq. (17) yields

$$\delta\lambda_m = -\varepsilon/2(\tfrac{1}{2} + 1/m\pi)\lambda_m = -\varepsilon/8(\tfrac{1}{2} + 1/m\pi)m^2\pi^2, \quad m = 1, 3, 5, \ldots$$

Similarly by using Eq. (14) we get

$$a_{mj} = 0, \quad m \neq j, \quad a_{mm} = -\varepsilon/4(\tfrac{1}{2} + 1/m\pi)$$

so

$$\delta Y_m = -\varepsilon/4(\tfrac{1}{2} + 1/m\pi)Y_m$$

Example B)—Sensitivity Analysis of Optimal Systems

The results of Secs. II and III may have application in the sensitivity study of the optimized systems governed by eigenproblems. A question of interest is to determine the variation in characteristic quantities due to the change of parameters and functions of a physical problem which are obtained in such a way as to optimize a certain response of the system. For illustration consider the problem of a column whose shape is determined such that among all the columns having the same length and volume has the highest buckling load. If x, λ, A, ϕ represent, respectively, the nondimensional axial coordinate, the buckling load, the variable area, and the bending moment then the governing boundary value problem for a clamped-crampel column is

$$\phi_{xx} + \lambda A^{-2}\phi = 0, \quad \begin{array}{l}\phi_x(0) - \phi_x(1) = 0 \\ \phi_\xi(0) + \phi(0) - \phi(1) = 0\end{array}$$

To achieve an optimum shape the following relation must hold[9] $\phi^2 = A^3$ and the resulting optimum shape comes out to be

$$A(x) = A_o \sin^2\theta(x)$$

with

$$\theta - \tfrac{1}{2}\sin 2\theta + \pi/2 = 2(\lambda/3)^{1/2} A_o^{-1} x$$

and

$$A_o = 1/\pi(\lambda/3)^{1/2}$$

the corresponding first buckling load and bending mode are

$$\lambda = 16\pi^2/3, \quad \phi(x) = A_o^{3/2}\sin^3\theta(x)$$

Now, let the optimum area be given a variation of the form

$$\delta A(x) = \varepsilon \sin \pi x$$

then, from Eq. (13) the resulting variation in the buckling load is found to be

$$\delta\lambda = -(4/\pi)\varepsilon\lambda$$

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Instability of Combustion

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THE following equations governing the phenomenon of intrinsic instability of combustion, leading to low frequency oscillations in a rocket motor using a single liquid propellant, were derived and investigated by L. Crocco[1–3]

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